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LETTER TO THE EDITOR

Intermittency in chaotic systems and Renyi entropies

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Abstract. We show that the Renyi entropies characterise the temporal intermittency in chaotic systems and are linked to a set of generalised Lyapounov exponents related to the time fluctuations of the responses to slight perturbations on the trajectory. It is also briefly indicated how a simple extension of a numerical algorithm proposed by Grassberger and Procaccia allows us to compute these entropies from a signal.

One of the more relevant problems in the numerical and experimental study of non-linear dynamical systems is the quantitative characterisation of chaotic signals. For example, the Kolmogorov entropy h and the spectrum of the Lyapounov exponents $\{\lambda_i\}$ provide a quantitative way of measuring how chaotic a system is. Indeed the Lyapounov exponents characterise the instabilities of nearby orbits while the Kolmogorov entropy gives a rough indication of the predictability time. Nevertheless one generally has variations of the chaoticity degree along a given trajectory and this intermittency can have a great relevance in some cases.

A classical example can be found in the Pomeau and Manneville (1980) mechanism for the onset of turbulence where bursts of strong chaoticity interrupt regular motion.

It is, however, clear that the Lyapounov exponents cannot measure the degree of intermittency because of their 'global' character. In a previous paper (Benzi *et al* 1985) we have therefore introduced a set of exponents $L(q)$ which generalise the maximal Lyapounov exponent λ_1 in order to give a quantitative description of this phenomenon. It is also shown (Benzi *et al* 1985) that the $L(q)$ can be regarded as a free energy $F(\beta)$ at the inverse temperature $\beta = 1 - q$ in the thermodynamical formalism for axiom A one-dimensional systems (Bowen 1975).

The purpose of this letter is to show that a more complete characterisation of intermittency is achieved by means of a set of generalised entropies K_q, first introduced by Renyi (1970). It is remarkable that these entropies can be computed also from an experimental signal with a simple extension of an algorithm recently proposed by Grassberger and Procaccia (1983) for the calculation of the K_q.

To be explicit let us consider a non-linear dynamical system with F degrees of freedom whose evolution $x(t)$ is given by

$$\dot{x} = f(x) \qquad f, x \in \mathbb{R}^F. \tag{1}$$

Our discussion can, however, be applied to an experimental signal or to the evolution given by a map as well.

Let us consider the discrete sequence

$$x_i = x(i\tau) \qquad i = 1, 2, \ldots, M \gg 1$$

and let $P_\varepsilon\ (i_1, \ldots i_d)$ be the probability that $\boldsymbol{x}_1$ fall in the hypercube (with edge ε) i_1, $\boldsymbol{x}_2$ in the hypercube i_2 and so on. The Renyi entropies are defined by

$$K_q = -\lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{d\to\infty}\frac{1}{\tau d(q-1)}\ln\left(\sum_{i_1\ldots i_d} P_\varepsilon(i_1\ldots i_d)^q\right) \tag{2}$$

and the Kolmogorov entropy is

$$h = -\lim_{\tau\to 0}\lim_{\varepsilon\to 0}\lim_{d\to\infty}\frac{1}{\tau d}\sum_{i_1\ldots i_d} P_\varepsilon(i_1\ldots i_d)\ln P_\varepsilon(i_1\ldots i_d). \tag{3}$$

It is trivial to see that $h = K_1 = \lim_{q\to 1} K_q$ and, moreover, to show, by general theorems of probability (Feller 1971), that K_q monotonically decreased with q. Our purpose is now to show how the Renyi entropies K_q characterise the temporal intermittency as well as the generalised Lyapounov exponents (introduced in Benzi *et al* (1985)) do.

Indeed non-constant Renyi entropies imply, in some sense, a multifractal structure (Benzi *et al* 1984) in the trajectory space, i.e. an anomalous scaling behaviour. We want to develop this idea in another context (Paladin *et al* 1986) since this letter is dedicated to showing how detailed information on the chaotic behaviour can be extracted by quantities which are accessible either in experiments or in numerical computations.

Let us therefore recall that we define the response R after a time τ to a perturbation acting at a time t as

$$R_t(\tau) = \frac{|\boldsymbol{\zeta}(t+\tau)|}{|\boldsymbol{\zeta}(t)|} \tag{4}$$

where $\boldsymbol{\zeta}(t)$ is given by the linearised evolution of system (1):

$$\dot{\boldsymbol{\zeta}} = (Df)\boldsymbol{\zeta} \qquad (Df)_{ij} = \partial f_i/\partial x_j. \tag{5}$$

The maximal Lyapounov exponent λ_1 is (Benettin *et al* 1980)

$$\lambda_1 = \lim_{\tau\to\infty}\tau^{-1}\langle\ln R_t(\tau)\rangle \tag{6}$$

where $\langle\ \rangle$ indicates a time average on the trajectory.

The exponents $L(q)$ are related to the moments of the response:

$$L(q) = \lim_{\tau\to\infty}\tau^{-1}\ln\langle R_t^q(\tau)\rangle. \tag{7}$$

Note that $\mathrm{d}L/\mathrm{d}q|_{q=c} = \lambda_1$. Moreover the degree of intermittency can be quantified by the deviation from the linear non-intermittent case $L(q) = \lambda_1 q$.

The extension of this idea to all the Lyapounov exponents is quite simple. Let us consider n tangent vectors $(1 \leq n \leq F)\boldsymbol{\zeta}^{(1)}, \ldots, \boldsymbol{\zeta}^{(n)}$ with different initial conditions and the same evolution equation (5); then we can define an n-order response as

$$R_t^{(n)}(\tau) = \frac{|\boldsymbol{\zeta}^{(1)}(t+\tau)\wedge\boldsymbol{\zeta}^{(2)}(t+\tau)\wedge\ldots\wedge\boldsymbol{\zeta}^{(n)}(t+\tau)|}{|\boldsymbol{\zeta}^{(1)}(t)\wedge\boldsymbol{\zeta}^{(2)}(t)\wedge\ldots\wedge\boldsymbol{\zeta}^{(n)}(t)|} \tag{8}$$

where $\wedge$ indicates the inner product.

Let us then recall (Benettin *et al* 1980) that the sum of the first n Lyapounov exponents is given by

$$l_n = \sum_{i=1}^{n}\lambda_i = \lim_{t\to\infty}\tau^{-1}\langle\ln R^{(n)}(\tau)\rangle \tag{9}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_F$. It is therefore natural to introduce the new exponents

$$L^{(n)}(q) = \lim_{t\to\infty} \tau^{-1} \ln\langle R_t^{(n)}(\tau)^q\rangle \tag{10}$$

whose properties are analogous to the $L(q)$ properties (i.e. $L^{(n)}(q)$ are concave functions for any n). Moreover we see that

$$\left.\frac{\mathrm{d}L^{(n)}}{\mathrm{d}q}\right|_{q=0} = \sum_{i=1}^{n} \lambda_i$$

and for non-intermittent behaviour

$$L^{(n)}(q) = \left(\sum_{i=1}^{n} \lambda_i\right) q.$$

Let us call n^* the number of non-negative Lyapounov exponents (i.e. $\lambda_{n^*} = 0$, $\lambda_{n^*+1} < 0$); the Pesin relation $h = \Sigma_{i=1}^{n^*} \lambda_i$ (Pesin 1976) can be written as

$$h = K_1 = \left.\frac{\mathrm{d}L^{(n^*)}}{\mathrm{d}q}\right|_{q=0}. \tag{11}$$

This relation is commonly believed to hold in almost all general systems and in the following we shall assume its validity.

We must stress that our definitions of Lyapounov exponents, Renyi entropies and so on have to be understood as referring to the natural measure. With this term we naively mean that the probability $P_\varepsilon(i_1 \ldots i_d)$ is deduced by the (chaotic) temporal evolution of the system.

Equation (11) is the analogue of the relation

$$\lambda_1 = \left.\frac{\mathrm{d}L}{\mathrm{d}q}\right|_{q=0}.$$

Moreover in the non-intermittent case:

$$L^{(n^*)}(q) = q l_{n^*} = qh \tag{12}$$

and the deviations from (12) give an indication of the degree of intermittency.

It is reasonable to look for a relation which links $L^{(n^*)}(q)$ to K_q. Let us indicate by $M_\varepsilon(\boldsymbol{x}(t), \Delta t)$ the 'quantity' of trajectories around $\boldsymbol{x}(t')$ which remain at a distance smaller than ε from $\boldsymbol{x}(t')$ for $t < t' < t + \Delta t$.

This naive definition can be made more precise without any difficulty (Grassberger 1986) but it is sufficient for our purposes.

We see that the M_ε decay exponentially in Δt because of the divergence of the n^*-dimensional volume of the tangent space ($F - n^*$ directions are in fact contracting).

For the time sequence notation $t = i\tau$ and $\Delta t = \tau d$, so that

$$M_\varepsilon(X(t), \Delta t) \propto \frac{1}{R_t^{(n^*)}(\Delta t)} n_i^{(1)}(\varepsilon) \tag{13}$$

where $n_i^{(1)}(\varepsilon)$ indicates the static part (at $\Delta t = 0$) and is defined as a density of points around $\boldsymbol{x}(t)$

$$n_i^{(1)}(\varepsilon) = \frac{1}{M-1} \sum_{j\neq i} \theta(\varepsilon - |\boldsymbol{x}_i - \boldsymbol{x}_j|). \tag{14}$$

The moments of (13) can be obtained by a time average (i.e. $(1/M)\sum_{i=1}^{M}(\cdot)$) on the trajectory:

$$\langle|M_\varepsilon(\Delta t)|^q\rangle \propto \exp(\tau d L^{(n^*)}(-q))\langle|M_\varepsilon(\Delta t=0)|^q\rangle. \tag{15}$$

The static part also has an anomalous scaling

$$\langle|M_\varepsilon(\Delta t=0)|^q\rangle = \frac{1}{M}\sum_{i=1}^{M}(n_i^{(1)}(\varepsilon))^q \propto \varepsilon^{\Phi(q)}$$

with $\Phi(q)$ defined in Paladin and Vulpiani (1984) and Benzi *et al* (1984).

It is perhaps interesting to note that $d_q=\Phi(q-1)/(q-1)$ are sometimes called Renyi dimensions and d_1 coincides with the information dimension while d_0 coincides with the fractal dimension.

The relation between $L^{(n^*)}(q)$ and K_q can now be easily achieved by noting that

$$M_\varepsilon(x(t),\Delta t) \propto P_\varepsilon(i_1,\ldots,i_d) \tag{16}$$

with i_1 centred around x_1, i_2 around x_2 and so on. Moreover $\sum_{i_1\ldots i_d} P_\varepsilon(i_1\ldots i_d)^q$ can be written in terms of $n_i^{(d)}(\varepsilon)$:

$$n_i^{(d)}(\varepsilon)=\frac{1}{M-1}\sum_{j\neq i}\theta(\varepsilon-|x_i^{(d)}-x_j^{(d)}|) \tag{17}$$

where $x_i^{(d)}=[x(i\tau), x(i\tau+\tau),\ldots,x(i\tau+(d-1)\tau)]\in\mathbb{R}^{Fd}$. Namely, one has

$$\sum_{i_1\ldots i_d} P_\varepsilon(i_1\ldots\ i_d)^{q+1}=\langle P_\varepsilon(i_1\ldots i_d)^q\rangle \propto \frac{1}{M}\sum_i[n_i^{(d)}(\varepsilon)]^q = c_d^{(q)}(\varepsilon). \tag{18}$$

Comparing (18), (16) and (15) with (2) we finally obtain

$$-L^{(n^*)}(-q)=qK_{q+1}. \tag{19}$$

It is worth remarking that $L^{(n)}(q)$, as defined in (10), can be measured only in numerical experiments while $L^{(n^*)}(q)$ (and therefore K_q) can be extracted by an (experimental) chaotic signal. The algorithm is a slight extension of a method introduced by Grassberger and Procaccia (1983) for the computation of K_2. This procedure allows us to avoid the use of box counting methods which are practically impossible whenever $F>3$. Equation (18) indeed shows that the numerical computation of $C_d^{(q)}(\varepsilon)$ requires the same computer time for each q value.

One can then extrapolate the corresponding Renyi entropy K_q by the limit

$$K_q=\lim_{\tau\to0}\lim_{\varepsilon\to0}\lim_{d\to\infty}\frac{1}{\tau(q-1)}\ln\left[\frac{C_d^{(q-1)}(\varepsilon)}{C_{d+1}^{(q-1)}(\varepsilon)}\right]. \tag{20}$$

We wish here to also recall that Cohen and Procaccia (1985) proposed an analogous method for computing the metric entropy h (in the limit $q\to1$) directly from

$$\tilde{C}_d(\varepsilon)=\frac{1}{M}\sum_{i=1}^{M}\ln(n_i^{(d)}(\varepsilon)). \tag{21}$$

Finally we note that all the quantities involved in (20) can also be obtained, at least in principle, by an experimental signal.

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